WAVES ON THE SURFACE OF A THIN LAYER OF VISCOUS LIQUID

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Under the assumption that the boundary layer approximation for the original equations is valid, we show the possibility of the existence of progressive waves on the surface of a vertically flowing film when surface tension is neglected. From the system of equations obtained for a thin layer of viscous liquid flowing down an inclined plane, one equation for perturbations of a thin film follows. Steady solutions of this equation allow periodic discontinuous solutions of the roll-wave type.

1. Wave-type flows of films have been investigated in [1-4]. A detailed survey of fundamental investigations is to be found in [3, 5].

In the present paper, under the assumption of the existence of quasisimple waves, an equation governing the propagation of perturbations on the surface of a film of viscous liquid is obtained by the method of integral relations. Steady solutions of the equation obtained allow discontinuous periodic solutions of the roll-wave type [5, 6]. The influence of surface tension on the structure of the roll waves is investigated.

In [3] it has been shown that when the thickness h of the film is less than the length of the wave under consideration, the following boundary layer approximation to the original equations is valid

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\sin\varphi+\nu\frac{\partial^2 u}{\partial y^2}-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1.1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1.2}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y}=-g\cos\varphi \tag{1.3}$$

The boundary condition on the rigid surface has the form

$$u(x,0)=v(x,0)=0 \tag{1.4}$$

while those on the surface of the film are

$$V_1=\frac{\partial h}{\partial t}+U_1\frac{\partial h}{\partial x} \tag{1.5}$$

$$\left(\frac{\partial u}{\partial y}\right)_{y=h}=0 \tag{1.6}$$

$$p=-\sigma\frac{\partial^2 h}{\partial x^2} \tag{1.7}$$

Here, μ is the longitudinal and v is the transverse velocity component, x, y are the longitudinal and transverse coordinates, φ is the angle of inclination of the surface, σ is the coefficient of surface tension (Fig. 1), ρ is the density of the liquid, g is the free-fall acceleration, and ν is the viscosity coefficient.

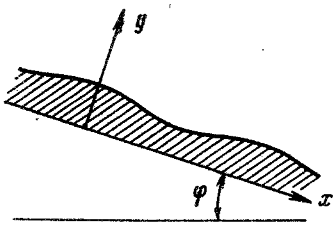

Fig. 1

Integrating Eqs. (1.1)-(1.3) across the layer from zero to the surface of the film and using relations (1.4)-(1.7), we obtain

$$\frac{\partial}{\partial t}\int_0^h u\,dy+\frac{\partial}{\partial x}\int_0^h u^2\,dy=-\nu\left(\frac{\partial u}{\partial y}\right)_{y=0}-gh\cos\varphi+\frac{\sigma h}{\rho}\frac{\partial^3 h}{\partial x^3} \tag{1.8}$$

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\int_0^h u\,dy=0 \tag{1.9}$$

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We prescribe the velocity profile in the form [1, 2, 4]

$$u = U_1 f(\eta), \quad \eta = y / h$$

where U_1 is the velocity at the surface of the film.

We introduce the notation

$$\alpha = \int_0^1 f(\eta)\, d\eta, \quad \gamma = \int_0^1 f^2(\eta)\, d\eta, \quad \beta = \left(\frac{\partial f}{\partial \eta}\right)_{\eta=0}$$

Equations (1.8), (1,9), written in terms of the mean flow velocity,

$$U = \int_0^h \frac{u}{h}\, dy = \alpha U_1$$

assume the form

$$\frac{\partial U}{\partial t} + \gamma_1 U \frac{\partial U}{\partial x} + \alpha_1 \frac{U^2}{h} \frac{\partial h}{\partial x} = -\frac{\nu\beta}{\alpha h^2} U - g \cos\varphi \frac{\partial h}{\partial x} + g \sin\varphi + \frac{\sigma}{\rho} \frac{\partial^3 h}{\partial x^3} \tag{1.10}$$

$$\frac{\partial h}{\partial t} + \frac{\partial U h}{\partial x} = 0$$

$$\left(\gamma_1 = \frac{2\gamma}{\alpha^2} - 1, \quad \alpha_1 = \frac{\gamma}{\alpha^2} - 1\right) \tag{1.11}$$

In [1, 2] and a number of other papers, the average over a cross section of the square of the velocity distribution function and the square of the mean value of this function is assumed to be equal in the derivation of the basic equations. The necessity for a valid averaging of Eq. (1.8) has been clearly demonstrated in [3]. It is established below that the existence of quasisimple waves on the surface of a film and steady solutions of the roll-wave type can be demonstrated, starting only from the equations as written in the form (1.10), (1.11).

The simplest form of the profile satisfying steady flow of the film is

$$u = U_1 \left(\frac{2y}{h} - \frac{y^2}{h^2}\right)$$

$$\alpha = {}^2/_3, \ \gamma = {}^8/_{15}, \ \gamma_1 = 1.4, \ \alpha_1 = 0.2, \ \beta / \alpha = 3 \tag{1.12}$$

Introducing perturbations of the velocity and the surface from their mean values by the relations

$$h = h_0 + h_+, \quad U = U_0 + u_+$$

and assuming that the coefficients in front of the viscosity term and the term that takes the surface tension into account are small, we obtain with an accuracy up to terms of second order in small quantities the following system of equations for h_+ and u_+

$$\frac{\partial u_+}{\partial t} + \gamma_1 U_0 \frac{\partial u_+}{\partial x} + \gamma_1 u_+ \frac{\partial u_+}{\partial x} + \frac{\alpha_1 U_0^2}{h_0^2}\left(\frac{\partial h_+}{\partial x} - h_+ \frac{\partial h_+}{\partial x}\right)$$

$$+ \frac{2U_0 u_+}{h_0} \frac{\partial h_+}{\partial x} + g \cos\varphi \frac{\partial h_+}{\partial x} = \frac{2\beta\nu}{\alpha h_0^3} U_0 h_+ - \frac{\beta\nu u_+}{h_0^2} + \frac{\sigma}{\rho} \frac{\partial^3 h_+}{\partial x^3}$$

$$\frac{\partial h_+}{\partial t} + U_0 \frac{\partial h_+}{\partial x} + u_+ \frac{\partial h_+}{\partial x} + h_0 \frac{\partial u_+}{\partial x} + h_+ \frac{\partial u_+}{\partial x} = 0 \tag{1.14}$$

In the derivation of (1.13) and (1.14) it was assumed that the averaged quantities are subject to the relationships for steady motion [2]. We shall make a number of comments on the system of equations (1.13), (1.14) we have obtained.

For $U_0 = 0$ and $\varphi = 0$, the system (1.13), (1.14) reduces to the shallow-water equations with friction at the bottom taken into account. In this system, progressive waves are possible – gravity waves on the surface of a heavy liquid of small depth.

For $\varphi = 90°$ and $U_0 \neq 0$ the system permits solutions in the form of progressive waves, the role of the term $g(\partial h/\partial x)$ being played by $\alpha_1 U_0^2 h_0^{-1} (\partial h/\partial x)$, even in the case of rather long waves when the effect of surface tension forces can be neglected.

In the system (1.13), (1.14), progressive waves exist when φ varies in a continuous manner.

Using Uizem's method [7], we shall seek a solution of the system (1.3), (1.4) in the form of a quasi-simple wave [8]

$$u_+ = ah_+ + bh_+^2 + c\frac{\partial^2 h}{\partial x^2} + D\int h_+ dx \tag{1.15}$$

The coefficients a, b, c, and D are determined from the condition that Eqs. (1.13) and (1.14) be identical. We assume that the coefficients a and b are of the order unity, while D and c, h_+ are of the order $\varepsilon \sim u_+/U_0$.

With an accuracy up to ε^2 an equation for h_+ follows from the system (1.13), (1.14):

$$\begin{gathered}
\frac{\partial h_+}{\partial t} + (U_0 + ah_0)\frac{\partial h_+}{\partial x} + 2(a + bh_0)\, h_+ \frac{\partial h_+}{\partial x} + Dh_0 h_+ ch_0 \frac{\partial^3 h_+}{\partial x^3} = 0 \\
a = \frac{U_0}{2h_0}(\gamma_1 - 1) + \left(\frac{U_0}{4h_0^2}(\gamma_1 - 1)^2 + \frac{\alpha_1 U_0^2}{h_0^2} + \frac{g\cos\varphi}{h_0}\right)^{1/2} \\
b = \left[(\gamma_1 - 2)a^2 + \frac{2U_0}{h_0}a - \frac{\alpha U_0^2}{h^2_0}\right][4ah_0 - 2(\gamma - 1)U_0]^{-1} \\
D = \left(\frac{\beta}{\alpha}\frac{\nu a}{h_0^2} - \frac{2\beta\nu U_0}{\alpha h_0^3}\right)[2ah_0 + U_0(1 - \gamma_1)]^{-1} \\
c = -\sigma[\rho(2h_0 a + (1 - \gamma)U_0)]^{-1}
\end{gathered} \tag{1.16}$$

For a film on a vertical wall

$$\begin{gathered}
\cos\varphi = 0,\ \ a = 0.7\ U_0 / h_0,\ \ b = 0.45\ U_0 / h_0^2 \\
D = -3.9\ \nu / h_0^3,\ \ c = -\sigma / \rho U_0
\end{gathered}$$

With the coefficients determined above, the differential relationship (1.15) that we have introduced transforms the system (1.13), (1.14) into two identical equations (1.16).

Thus, the propagation of an arbitrary perturbation on the surface of a thin film is governed by the Korteweg–de Vries equation with a low-frequency boosting energy

$$\frac{\partial h_+}{\partial t} + 1.7U_0\frac{\partial h_+}{\partial x} + 2.3U_0\frac{h_+}{h_0}\frac{\partial h}{\partial x} + \frac{\sigma h_0}{\rho U_0}\frac{\partial^3 h}{\partial x^3} = \frac{3.9\nu}{h_0^2}h_+ \tag{1.17}$$

The right side of Eq. (1.17) is responsible for the growth of momentum and energy with time in the wave under consideration.

2. Infinite growth of the amplitude is prevented by the nonlinear term, which "upsets" the perturbation. A discontinuity can occur, with surface tension inhibiting its formation. We shall exclude from consideration the influence of surface tension on the process of wave formation. Then Eq. (1.17) has discontinuous periodic solutions.

We introduce the self-similar coordinate $\xi = x - Ct$ into (1.17).

$$-C\frac{\partial h_+}{\partial \xi} + 1.7U_0\frac{\partial h_+}{\partial \xi} + 2.3U_0\frac{h_+}{h_0}\frac{\partial h_+}{\partial \xi} + \frac{\sigma h_0}{\rho U_0}\frac{\partial^3 h_+}{\partial \xi^3} = \frac{3.9\nu}{h_0^2}h_+ \tag{2.1}$$

In the case where the third derivative with respect to ξ is absent in Eq. (2.1) we have

$$-C\frac{\partial h_1}{\partial \xi} + 1.7U_0\frac{\partial h_1}{\partial \xi} + 2.3U_0\frac{h_1}{h_0}\frac{\partial h_1}{\partial \xi} = \frac{3.9\nu h_1}{h_0} \tag{2.2}$$

Discontinuous periodic solutions of this equation are investigated in [9, 10].

In [10] Eq. (2.2) is investigated as a model equation, governing a system of roll waves. Introducing $F = 2.3\,U_0 h_0^{-1} h_1$, we have the equation

$$-\omega\frac{\partial F}{\partial x} + F\frac{\partial F}{\partial x} = \frac{3.9\nu}{h_0^2}F \tag{2.3}$$

$$\omega = 1.7U_0 - C \tag{2.4}$$

Equation (2.3) has been studied in [9, 10]. According to results of these papers Eq. (2.3) has discontinuous periodic solutions for $\omega = 0$, and it follows from (2.3) that the propagation speed of the roll-wave $C = 1.7\,U_0$, which coincides with the propagation speed of a sufficiently long perturbation (weak discontinuity) as follows from Eq. (1.17). This value is approximately equal to the lowest wave speed on the surface of a film according to the calculations of [4].

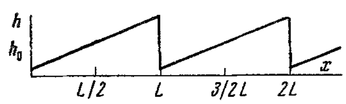

Fig. 2

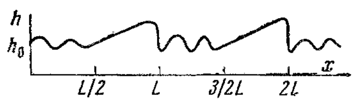

Fig. 3

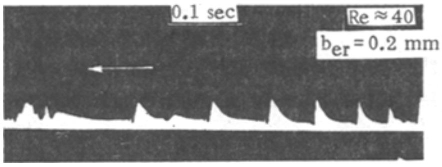

Fig. 4

Prescribing the wavelength L and using relations of the Rankine–Hugoniot type [9] at the discontinuities, we obtain an expression for the thickness of the layer in the region where function h_1 is continuous.

$$h_1 = \frac{1.7}{\mathrm{Re}}\left(x - \frac{L}{2}\right) \qquad \left(\mathrm{Re} = \frac{U_0 h_0}{\nu}\right) \tag{2.5}$$

A picture of the surface of the film according to (2.5) is shown in Fig. 2.

The solution (2.5) satisfies the condition: the energy of the wave up to the discontinuity is greater than it is after the discontinuity [9].

Thus the possibility of the existence of a "noncapillary wave" on the surface of a vertical film is demonstrated.

We shall estimate the influence of surface tension on the steady pattern of a roll wave. Proceeding as in the investigation of shock waves [8], we represent the general solution in the interval of continuity in the form

$$h_+ = h_1 + f \tag{2.6}$$

where h_1 is given by Fq. (2.5) and f is a small correction, taking surface tension into account. Inserting (2.6) into (1.17), written in terms of the self-similar variable $\xi = x - Ct$, and linearizing, we have the equation for f

$$\eta \frac{\partial f}{\partial \eta} - k \frac{\partial^3 f}{\partial \eta^3} = 0$$

$$\left(\eta = \frac{x}{L} - \frac{1}{2}, \quad k = \frac{1}{3.9}\frac{1}{\mathrm{We}}\mathrm{Re}\frac{h_0^2}{L^2}, \quad \mathrm{We} = \frac{\sigma}{\rho U_0^2 L}\right) \tag{2.7}$$

Introducing

$$\eta_1 = \eta k^{-1/3}, \quad \Phi = \partial f / \partial \eta_1$$

we reduce Eq. (2.7) to Airy's equation [11]

$$\partial^2\Phi/\partial\eta_1^2 - \eta_1\Phi = 0 \tag{2.8}$$

The origin $\eta_1 = 0$ is at the center of the region of continuity with $h = h_0$. It is known [11] that Eq. (2.8) has oscillatory solutions for $\eta_1 < 0$ and monotonic ones for $\eta_1 > 0$. Analyzing solutions of (2.7) on the basis of the known solution of (2.8) [11] in the region of continuity, we can state that qualitatively a steady wave on the surface of the film must have the shape shown in Fig. 3.

Figure 4 shows the shape of the surface of a film recorded by the shadow method during the latter's flow along the surface of a vertical tube of large diameter at Re ≈ 40. Such a pattern has been observed in many experiments connected with the investigation of wave formation on films [5].

Comparing experiment with theory, one can speak only of a qualitative similarity of the processes and of an approximate periodicity of the wave process.

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